ECE 604, Lecture 29

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Contents

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1 Uniqueness Theorem

In this section, we will prove uniqueness theorem for electrodynamic problems. Assume that there exist two solutions in the presence of one set of common impressed sources $\mathbf{J}_i$ and $\mathbf{M}_i$. Namely, these two solutions are $\mathbf{E}^a$, $\mathbf{H}^a$, $\mathbf{E}^b$, $\mathbf{H}^b$. Both of them satisfy Maxwell's equations and the same boundary conditions. Then, considering general anisotropic inhomogeneous media, where the tensors $\overline{\boldsymbol{\mu}}$ and $\overline{\varepsilon}$ can be complex so that lossy media can be included, it follows that

$$\nabla \times \mathbf{E}^a = -j\omega\overline{\boldsymbol{\mu}} \cdot \mathbf{H}^a - \mathbf{M}_i \tag{1.1}$$

$$\nabla \times \mathbf{E}^b = -j\omega\overline{\boldsymbol{\mu}} \cdot \mathbf{H}^b - \mathbf{M}_i \tag{1.2}$$

$$\nabla \times \mathbf{H}^a = j\omega\overline{\varepsilon} \cdot \mathbf{E}^a + \mathbf{J}_i \tag{1.3}$$

$$\nabla \times \mathbf{H}^b = j\omega\overline{\varepsilon} \cdot \mathbf{E}^b + \mathbf{J}_i \tag{1.4}$$

By taking the difference of these two solutions, we have

$$\nabla \times (\mathbf{E}^a - \mathbf{E}^b) = -j\omega\overline{\boldsymbol{\mu}} \cdot (\mathbf{H}^a - \mathbf{H}^b) \tag{1.5}$$

$$\nabla \times (\mathbf{H}^a - \mathbf{H}^b) = j\omega\overline{\varepsilon} \cdot (\mathbf{E}^a - \mathbf{E}^b) \tag{1.6}$$

Or alternatively, defining $\delta\mathbf{E} = \mathbf{E}^a - \mathbf{E}^b$ and $\delta\mathbf{H} = \mathbf{H}^a - \mathbf{H}^b$, we have

$$\nabla \times \delta\mathbf{E} = -j\omega\overline{\boldsymbol{\mu}} \cdot \delta\mathbf{H} \tag{1.7}$$

$$\nabla \times \delta\mathbf{H} = j\omega\overline{\varepsilon} \cdot \delta\mathbf{E} \tag{1.8}$$

The difference solutions satisfy the original source-free Maxwell's equations.

By taking the left dot product of $\delta\mathbf{H}^*$ with (1.7), and then the left dot product of $\delta\mathbf{E}^*$ with the complex conjugation of (1.8), we obtain

$$\begin{aligned} \delta\mathbf{H}^* \cdot \nabla \times \delta\mathbf{E} &= -j\omega\delta\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \delta\mathbf{H} \\ \delta\mathbf{E} \cdot \nabla \times \delta\mathbf{H}^* &= -j\omega\delta\mathbf{E} \cdot \overline{\varepsilon}^* \cdot \delta\mathbf{E}^* \end{aligned} \tag{1.9}$$

Now, taking the difference of the above, we get

$$\begin{aligned} \delta\mathbf{H}^* \cdot \nabla \times \delta\mathbf{E} - \delta\mathbf{E} \cdot \nabla \times \delta\mathbf{H}^* &= \nabla \cdot (\delta\mathbf{E} \times \delta\mathbf{H}^*) \\ &= -j\omega\delta\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \delta\mathbf{H} + j\omega\delta\mathbf{E} \cdot \overline{\varepsilon}^* \cdot \delta\mathbf{E}^* \end{aligned} \tag{1.10}$$

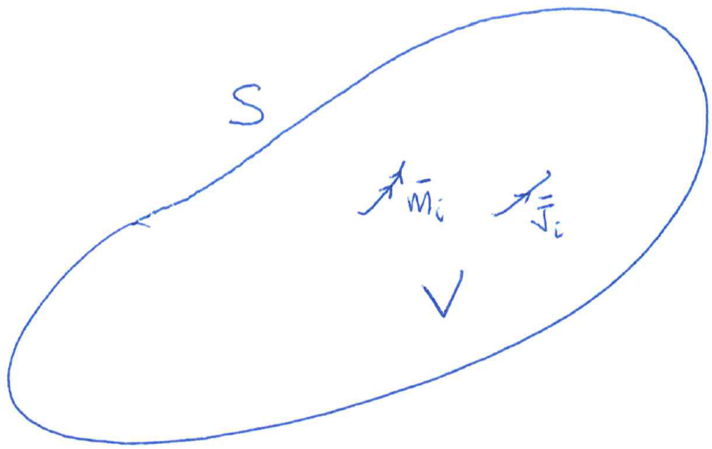

Figure 1:

Next, integrating the above equation over a volume V bounded by a surface S as shown in Figure 1. Two scenarios are possible: one that the volume V contains the impressed sources, and two, that the sources are outside the volume V. After making use of Gauss' divergence theorem, we arrive at

$$\begin{aligned}\iint_V \nabla\cdot(\delta\mathbf{E}\times\delta\mathbf{H}^*)dV &= \oiint_S(\delta\mathbf{E}\times\delta\mathbf{H}^*)\cdot d\mathbf{S}\\ &= \iiint_V[-j\omega\delta\mathbf{H}^*\cdot\overline{\boldsymbol{\mu}}\cdot\delta\mathbf{H}+j\omega\delta\mathbf{E}\cdot\overline{\boldsymbol{\varepsilon}}^*\cdot\delta\mathbf{E}^*]dV \end{aligned}\quad (1.11)$$

And next, we would like to know the kind of boundary conditions that would make the left-hand side equal to zero. It is seen that the surface integral on the left-hand side will be zero if:[1]

1. If $\hat{n}\times\mathbf{E}$ is specified over S so that $\hat{n}\times\mathbf{E}_a=\hat{n}\times\mathbf{E}_b$, then $\hat{n}\times\delta\mathbf{E}=0$ or the PEC boundary condition for $\delta\mathbf{E}$, and then[2]

$\oiint_S(\delta\mathbf{E}\times\delta\mathbf{H}^*)\cdot\hat{n}dS=\oiint_S(\hat{n}\times\delta\mathbf{E})\cdot\delta\mathbf{H}^*dS=0.$

2. If $\hat{n}\times\mathbf{H}$ is specified over S so that $\hat{n}\times\mathbf{H}_a=\hat{n}\times\mathbf{H}_b$, then $\hat{n}\times\delta\mathbf{H}=0$ or the PMC boundary condition for $\delta\mathbf{H}$, and then

[1] In the following, please be reminded that PEC stands for "perfect electric conductor", while PMC stands for "perfect magnetic conductor". PMC is the dual of PEC. Also, a fourth case of impedance boundary condition is possible, which is beyond the scope of this course. Interested readers may consult Chew, Theory of Microwave and Optical Waveguides.

[2] Using the vector identity that $\mathbf{a}\cdot(\mathbf{b}\times c)=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$.

$\oiint_S (\delta\mathbf{E} \times \delta\mathbf{H}^*) \cdot \hat{n} dS = -\oiint_S (\hat{n} \times \delta\mathbf{H}^*) \cdot \delta\mathbf{E} dS = 0.$

3. If $\hat{n}\times\mathbf{E}$ is specified over S_1, and $\hat{n}\times\mathbf{H}$ is specified over S_2 (where $S_1 \cup S_2 = S$), then $\hat{n}\times\delta\mathbf{E} = 0$ (PEC boundary condition) on S_1, and $\hat{n}\times\delta\mathbf{H} = 0$ (PMC boundary condition) on S_2, then the left-hand side becomes

$$\oiint_S (\delta\mathbf{E} \times \delta\mathbf{H}^*) \cdot \hat{n} dS = \iint_{S_1} + \iint_{S_2} = \iint_{S_1} (\hat{n} \times \delta\mathbf{E}) \cdot \delta\mathbf{H}^* dS - \iint_{S_2} (\hat{n} \times \delta\mathbf{H}^*) \cdot \delta\mathbf{E} dS = 0.$$

Thus, under the above three scenarios, the left-hand side of (1.11) is zero, and then the right-hand side of (1.11) becomes

$$\iiint_V [-j\omega\delta\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \delta\mathbf{H} + j\omega\delta\mathbf{E} \cdot \overline{\boldsymbol{\varepsilon}}^* \cdot \delta\mathbf{E}^*] dV = 0 \tag{1.12}$$

For lossless media, $\overline{\boldsymbol{\mu}}$ and $\overline{\boldsymbol{\varepsilon}}$ are hermitian tensors (or matrices[3]), then it can be seen, using the properties of hermitian matrices or tensors, that $\delta\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \delta\mathbf{H}$ and $\delta\mathbf{E} \cdot \overline{\boldsymbol{\varepsilon}}^* \cdot \delta\mathbf{E}^*$ are purely real. Taking the imaginary part of the above equation yields

$$\iiint_V [-\delta\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \delta\mathbf{H} + \delta\mathbf{E} \cdot \overline{\boldsymbol{\varepsilon}}^* \cdot \delta\mathbf{E}^*] dV = 0 \tag{1.13}$$

The above two terms correspond to stored magnetic field energy and stored electric field energy in the difference solutions $\delta\mathbf{H}$ and $\delta\mathbf{E}$, respectively. The above being zero does not imply that $\delta\mathbf{H}$ and $\delta\mathbf{E}$ are zero.

For resonance solutions, the stored electric energy can balance the stored magnetic energy. The above resonance solutions are those of the difference solutions satisfying PEC or PMC boundary condition or mixture thereof. Therefore, $\delta\mathbf{H}$ and $\delta\mathbf{E}$ need not be zero, even though (1.13) is zero. This happens when we encounter solutions that are the resonant modes of the volume V bounded by surface S.

Uniqueness can only be guaranteed if the medium is lossy as shall be shown later. It is also guaranteed if lossy impedance boundary conditions are imposed.[4] First we begin with the isotropic case.

1.1 Isotropic Case

It is easier to see this for lossy isotropic media. Then (1.12) simplifies to

$$\iiint_V [-j\omega\mu|\delta\mathbf{H}|^2 + j\omega\varepsilon^*|\delta\mathbf{E}|^2] dV = 0 \tag{1.14}$$

For isotropic lossy media, $\mu = \mu' - j\mu''$ and $\varepsilon = \varepsilon' - j\varepsilon''$. Taking the real part of the above, we have from (1.14) that

$$\iiint_V [-\omega\mu''|\delta\mathbf{H}|^2 - \omega\varepsilon''|\delta\mathbf{E}|^2] dV = 0 \tag{1.15}$$

[3] Tensors are a special kind of matrices.

[4] See Chew, Theory of Microwave and Optical Waveguides.

Since the integrand in the above is always negative definite, the integral can be zero only if

$$\delta \mathbf{E} = 0, \quad \delta \mathbf{H} = 0 \tag{1.16}$$

everywhere in V, implying that $\mathbf{E}_a = \mathbf{E}_b$, and that $\mathbf{H}_a = \mathbf{H}_b$. Hence, it is seen that uniqueness is guaranteed only if the medium is lossy. The physical reason is that when the medium is lossy, a pure time-harmonic solution cannot exist due to loss. The modes, which are the source-free solutions of Maxwell's equations, are decaying sinusoids.

Notice that the same conclusion can be drawn if we make μ'' and ε'' negative. This corresponds to active media, and uniqueness can be guaranteed for a time-harmonic solution. In this case, no time-harmonic solution exists, and the resonant solution is a growing sinusoid.

1.2 General Anisotropic Case

The proof for general anisotropic media is more complicated. For the lossless anisotropic media, we see that (1.12) is purely imaginary. However, when the medium is lossy, this same equation will have a real part. Hence, we need to find the real part of (1.12) for the general lossy case.

1.2.1 About taking the Real and Imaginary Parts of a Complex Expression

To this end, we digress on taking the real and imaginary parts of a complex expression. Here, we need to find the complex conjugate[5] of (1.12), which is scalar, and add it to itself to get its real part. The complex conjugate of the scalar

$$c = \delta \mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \delta \mathbf{H}$$

is[6]

$$c^* = \delta \mathbf{H} \cdot \overline{\boldsymbol{\mu}}^* \cdot \delta \mathbf{H}^* = \delta \mathbf{H}^* \cdot \overline{\boldsymbol{\mu}}^\dagger \cdot \delta \mathbf{H}$$

Similarly, the complex conjugate of the scalar

$$d = \delta \mathbf{E} \cdot \overline{\boldsymbol{\varepsilon}}^* \cdot \delta \mathbf{E}^* = \delta \mathbf{E}^* \cdot \overline{\boldsymbol{\varepsilon}}^\dagger \cdot \delta \mathbf{E}$$

is

$$d^* = \delta \mathbf{E}^* \cdot \overline{\boldsymbol{\varepsilon}}^\dagger \cdot \delta \mathbf{E}$$

Therefore,

$$\Im m \left(\delta \mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \delta \mathbf{H} \right) = \frac{1}{2j} \delta \mathbf{H}^* \cdot \left(\overline{\boldsymbol{\mu}} - \overline{\mu}^\dagger \right) \cdot \delta \mathbf{H}$$

[5] Also called hermitian conjugate.

[6] To arrive at these expressions, one makes use of the matrix algebra rule that if $\overline{\mathbf{D}} = \overline{\mathbf{A}} \cdot \overline{\mathbf{B}} \cdot \overline{\mathbf{C}}$, then $\overline{\mathbf{D}}^t = \overline{\mathbf{C}}^t \cdot \overline{\mathbf{B}}^t \cdot \overline{\mathbf{A}}^t$. This is true even for non-square matrices. But for our case here, $\overline{\mathbf{A}}$ is a 1×3 row vector, and $\overline{\mathbf{C}}$ is a 3×1 column vector, and $\overline{\mathbf{B}}$ is a 3×3 matrix. In vector algebra, the transpose of a vector is implied. Also, in our case here, $\overline{\mathbf{D}}$ is a scalar, and hence, its transpose is itself.

$$\Im m \left(\delta \mathbf{E} \cdot \overline{\varepsilon} \cdot \delta \mathbf{E}^*\right) = \frac{1}{2j} \delta \mathbf{E}^* \cdot \left(\overline{\varepsilon} - \overline{\varepsilon}^\dagger\right) \cdot \delta \mathbf{E}$$

and similarly for the real part.

Finally, after taking the complex conjugate of the scalar quantity (1.12) and adding it to itself, we have

$$\iiint_V [-j\omega \delta \mathbf{H}^* \cdot (\overline{\boldsymbol{\mu}} - \overline{\boldsymbol{\mu}}^\dagger) \cdot \delta \mathbf{H} - j\omega \delta \mathbf{E}^* \cdot (\overline{\varepsilon} - \overline{\varepsilon}^\dagger) \cdot \delta \mathbf{E}] dV = 0 \tag{1.17}$$

For lossy media, $-j(\overline{\boldsymbol{\mu}} - \overline{\boldsymbol{\mu}}^\dagger)$ and $-j(\overline{\varepsilon} - \overline{\varepsilon}^\dagger)$ are hermitian negative matrices. Hence the integrand is always negative definite, and the above equation cannot be satisfied unless $\delta \mathbf{H} = \delta \mathbf{E} = 0$ everywhere in V. Thus, uniqueness is guaranteed in a lossy anisotropic medium.

Similar statement can be made as the isotropic case if the medium is active. Then the integrand is positive definite, and the above equation cannot be satisfied unless $\delta \mathbf{H} = \delta \mathbf{E} = 0$ everywhere in V and hence, uniqueness is satisfied.

1.3 Hind Sight

The proof of uniqueness for Maxwell's equations is very similar to the proof of uniqueness for a matrix equation

$$\overline{\mathbf{A}} \cdot \mathbf{x} = \mathbf{b} \tag{1.18}$$

If a solution to a matrix equation exists without excitation, namely, when $\mathbf{b} = 0$, then the solution is the null space solution, namely, $\mathbf{x} = \mathbf{x}_N$. In other words,

$$\overline{\mathbf{A}} \cdot \mathbf{x}_N = 0 \tag{1.19}$$

These null space solutions exist without a "driving term" $\mathbf{b}$ on the right-hand side. For Maxwell's Equations, $\mathbf{b}$ corresponds to the source terms. They are like the homogeneous solution of an ordinary differential equation or a partial differential equation. In an enclosed region of volume V bounded by a surface S, homogeneous solutions are the resonant solutions of this Maxwellian system. When these solutions exist, they give rise to non-uniqueness.

Also, notice that (1.7) and (1.8) are Maxwell's equations without the source terms. In a closed region V bounded by a surface S, only resonance solutions for $\delta \mathbf{E}$ and $\delta \mathbf{H}$ with the relevant boundary conditions can exist when there are no source terms.

As previoulsy mentioned, one way to ensure that these resonant solutions do not exist is to put in loss or gain. When loss or gain is present, then the resonant solutions are decaying sinusoids or growing sinusoids. Since we are looking for solutions in the frequency domain, or time harmonic solutions, these non-sinusoidal solutions are outside the solution space: They are not part of the time-harmonic solutions we are looking for. Therefore, there are no resonant null-space solutions.

1.4 Connection to Poles of a Linear System

The output to input of a linear system can be represented by a transfer function $H(\omega)$. If $H(\omega)$ has poles, and if the system is lossless, the poles are on the real axis. Therefore, when $\omega = \omega_{\text{pole}}$, the function $H(\omega)$ becomes undefined. This also gives rise to non-uniqueness of the output with respect to the input. Poles usually correspond to resonant solutions, and hence, the non-uniqueness of the solution is intimately related to the non-uniqueness of Maxwell's equations at the resonant frequencies of a structure. This is illustrated in the upper part of Figure 2.

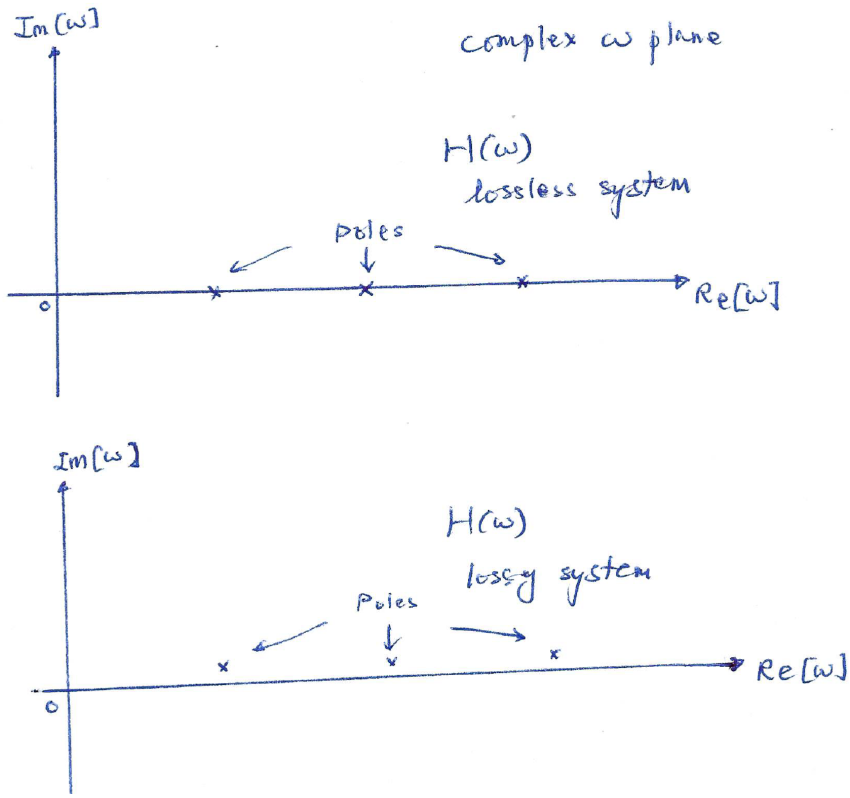

Figure 2:

However, if loss is introduced, these poles will move away from the real axis as shown in the lower part of Figure 2. Then the transfer function is uniquely determined for all frequencies, and uniqueness of the solutio is guaranteed.

1.5 Radiation from Antenna Sources

The above uniqueness theorem guarantees that if we have some antennas with prescribed current sources on them, the radiated field from these antennas are unique. To see how this can come about, we first study the radiation of sources into a region V bounded by a large surface S_{inf} as shown in Figure 3.

Even when $\hat{n} \times \mathbf{E}$ or $\hat{n} \times \mathbf{H}$ are specified on the surface at S_{inf}, the solution is nonunique because the volume V bounded by S_{inf}, can have many resonant solutions. In fact, the region will be replete with resonant solutions as one makes S_{inf} become very large. The way to remove these resonant solutions is to introduce an infinitesimal amount of loss in region V. Then these resonant solutions will disappear. Now we can take S_{inf} to infinity, and the solution will always be unique.

Notice that if $S_{\text{inf}} \to \infty$, the waves that leave the sources will never be reflected back because of the small amount of loss. The radiated field will just disappear into infinity. This is just what radiation loss is: power that propagate to infinity, but never to return. In fact, one way of guaranteeing the uniqueness of the solution in region V when S_{inf} is infinitely large, or that V is infinitely large is to impose the radiation condition: the waves that radiate to infinity is an outgoing wave only, and never do they return. This is also called the Sommerfeld radiation condition. Uniqueness of the field outside the sources is always guaranteed if we assume that the field radiates to infinity and never to return.

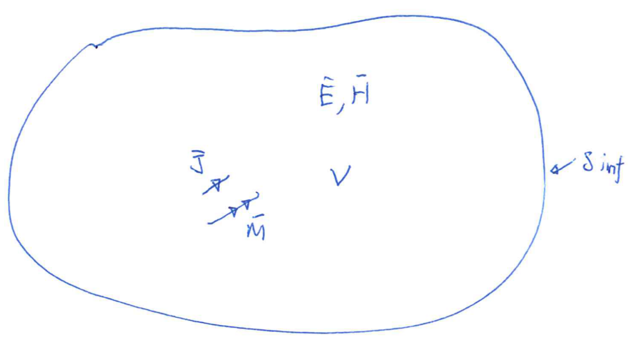

Figure 3: